



$$2mu''_1 + 4ku_1 + 3k(u_2 - u_1) = 0$$

$$5mu''_2 + ku_2 + 3k(u_1 - u_2) = 0$$

$$\begin{cases} 2u''_1 + 7u_1 - 3u_2 = 0 \\ 5u''_2 - 3u_1 + 4u_2 = 0 \end{cases}$$

$$(2\omega_0^2 - \omega^2)A - 3\omega_0^2 B = 0$$

$$-3\omega_0^2 A + (4\omega_0^2 - 5\omega^2)B = 0$$

$$(2\omega_0^2 - \omega^2)(4\omega_0^2 - 5\omega^2) - 9\omega_0^4 = 0$$

$$10\omega^4 - 43\omega_0^2\omega^2 + 19\omega_0^4 = 0$$

$$A = 43^2 - 260 = 1089 = (33)^2$$

$$\omega^2 = \frac{43 \pm 33}{20} \quad \begin{matrix} \nearrow \frac{76}{20} = \frac{38}{10} = \frac{19}{5} \\ \searrow \frac{10}{20} = \frac{1}{2} \end{matrix}$$

$$\omega^2 = \frac{1}{2}\omega_0^2 \Rightarrow 6A - 3B = 0 \quad 2A - B = 0 \quad \begin{cases} A=1 \\ B=2 \end{cases}$$

$$\omega^2 = \frac{19}{5}\omega_0^2 \Rightarrow -3A - 15B = 0 \quad 18A + 15B = 0 \quad \begin{cases} A = -5 \\ B = 1 \end{cases}$$

$$x_1 = R_1 \cos\left(\frac{\omega_0 t}{\sqrt{2}} + \phi_1\right) + \cancel{R_2} \sqrt{2} \cos\left(\sqrt{\frac{19}{5}}\omega_0 t + \phi_2\right)$$

$$x_2 = 2R_2 \cos\left(\frac{\omega_0 t}{\sqrt{2}} + \phi_1\right) - \cancel{R_1} \sqrt{2} \cos\left(\sqrt{\frac{19}{5}}\omega_0 t + \phi_2\right)$$

so $h(0)$, $h(-l)$ et $h(l)$

$$h(0) = h_0 \quad h(-l) = h_0 e^{+\frac{2\ell h_0}{l}} = h_0 e^{\frac{2h_0 l}{l}} = 4h_0$$

$$h(l) = h_0 e^{\frac{2h_0 l}{l}} = 4h_0$$

so EDP

$$\rho S \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial u} (E S \frac{\partial u}{\partial x}) = E \left[\frac{\partial S}{\partial x} \cdot \frac{\partial u}{\partial x} + S \frac{\partial^2 u}{\partial x^2} \right]$$

$$\text{for } -l \leq x \leq 0 \quad S(x) = b h_0 e^{-\frac{2h_0 x}{l}}$$

$$\frac{\partial S}{\partial x} = -\frac{2h_0}{l} S(x)$$

$$\rho \frac{\partial^2 u_1}{\partial t^2} = E \left[-\frac{2h_0}{l} S \frac{\partial u_1}{\partial x} + S \frac{\partial^2 u_1}{\partial x^2} \right]$$

$$\textcircled{1} \quad \rho \frac{\partial^2 u_1}{\partial t^2} = E \left[\frac{\partial u_1}{\partial x^2} - \frac{2h_0}{l} S \frac{\partial u_1}{\partial x} \right]$$

$$\text{for } 0 \leq x \leq l \quad S(x) = b h_0 e^{\frac{2h_0 x}{l}}$$

$$\frac{\partial S}{\partial x} = \frac{2h_0}{l} S(x)$$

$$\rho \frac{\partial^2 u_2}{\partial t^2} = E \left[\frac{2h_0}{l} S \frac{\partial u_2}{\partial x} + S \frac{\partial^2 u_2}{\partial x^2} \right]$$

$$\textcircled{2} \quad \rho \frac{\partial^2 u_2}{\partial t^2} = E \left[\frac{\partial u_2}{\partial x^2} + \frac{2h_0}{l} S \frac{\partial u_2}{\partial x} \right]$$

so variables séparées.

$$u_1(x, t) = X_1(x) \cdot q(t)$$

$\hat{q}(t)$ car $q(t)$ non nulle

$$u_2(x, t) = X_2(x) \cdot q(t)$$

$$\textcircled{1} \Rightarrow \rho X_1 q'' = E \left[X_1'' q - \frac{2h_0}{l} X_1' q \right]$$

$$\textcircled{2} \Rightarrow \rho X_2 q'' = E \left[X_2'' q + \frac{2h_0}{l} X_2' q \right]$$

$$\Rightarrow \frac{q''}{q} = \frac{E \left[X_1'' - \frac{2h_0}{l} X_1' \right]}{\rho X_1} = \frac{E \left[X_2'' + \frac{2h_0}{l} X_2' \right]}{\rho X_2} = -\omega^2$$

sous les conditions

$$q'' + \omega^2 q = 0 \Rightarrow q = \alpha \cos \omega t + \beta \sin \omega t.$$

$$X_1'' - \frac{2\ell^2}{E} X_1' + \omega^2 \frac{\rho}{E} X_1 = 0$$

$$X_2'' + \frac{2\ell^2}{E} X_2' + \omega^2 \frac{\rho}{E} X_2 = 0$$

avec $k^2 = \omega^2 \frac{\rho}{E}$

$$X_1'' - \frac{2k^2}{\ell} X_1' + k^2 X_1 = 0$$

$$X_2'' + \frac{2k^2}{\ell} X_2' + k^2 X_2 = 0$$

les polynômes caractéristiques et solutions

$$\textcircled{1} \quad r^2 - \frac{2k^2}{\ell} r + k^2 = 0$$

$$\Delta' = \left(\frac{k^2}{\ell}\right)^2 - k^2 < 0 \quad \text{si } k \ell > k^2$$

$$\textcircled{2} \quad r^2 + \frac{2k^2}{\ell} r + k^2 = 0$$

$$\Delta' = \left(\frac{k^2}{\ell}\right)^2 - k^2 < 0 \quad \text{même}$$

$$\Delta' = -\left[k^2 - \left(\frac{k^2}{\ell}\right)^2\right] = -\alpha^2$$

$$\textcircled{3} \Rightarrow r = \frac{k^2}{\ell} \pm i\alpha$$

$$\textcircled{4} \Rightarrow r = -\frac{k^2}{\ell} \pm i\alpha$$

$$X_1(x) = e^{\frac{k^2}{\ell}x} \left[A_1 \cos \alpha x + B_1 \sin \alpha x \right]$$

$$X_2(x) = e^{-\frac{k^2}{\ell}x} \left[A_2 \cos \alpha x + B_2 \sin \alpha x \right]$$

5. Ch. der Ph.

$$u_1(-l, t) = 0 \Rightarrow X_1(l) \cdot q(t) = 0 \Rightarrow X_1(l) = 0$$

$$u_1(\sigma t) = u_2(\sigma t) \Rightarrow X_1(0) \cdot q(t) = X_2(0) \cdot q(t) \Rightarrow X_1(0) = X_2(0)$$

$$N_1(\sigma t) = N_2(\sigma t) \Rightarrow \bar{E}S_1(0) \cdot X'_1(0) = \bar{E}S_2(0)X'_2(0) \Rightarrow X'_1(0) = X'_2(0)$$

$$u_2(l, t) = 0 \Rightarrow X_2(l) \cdot q(t) = 0 \Rightarrow X_2(l) = 0$$

$$X_1(-l) = 0 \Rightarrow e^{-\frac{\sigma l}{2}} [A_1 \cos \omega t - B_1 \sin \omega t] = 0$$

$$X_1(0) = X_2(0) \Rightarrow A_1 = A_2$$

$$X'_1(0) = X'_2(0) \Rightarrow \frac{\hbar^2}{l} A_1 + \alpha B_1 = -\frac{\hbar^2}{l} A_2 + \alpha B_2$$

$$X_2(l) = 0 \Rightarrow e^{-\frac{\sigma l}{2}} [A_2 \cos \omega t + B_2 \sin \omega t] = 0$$

$$\begin{cases} A_2 = A_1 \\ A_1 \cos \omega t - B_1 \sin \omega t = 0 \\ A_2 \cos \omega t + B_2 \sin \omega t = 0 \\ \frac{\hbar^2}{l} A_1 + \alpha B_1 - \alpha B_2 = 0 \end{cases}$$

rekt. lin. homogenes do 3 dg or 3 zwc \Rightarrow sol. triviale
wegen $\Delta \det = 0$

$$\begin{vmatrix} \cos \omega t & -\sin \omega t & 0 \\ \sin \omega t & 0 & \sin \omega t \\ \frac{\hbar^2}{l} & \alpha & -\alpha \end{vmatrix} = 0$$

$$-\frac{\hbar^2}{l} \sin^2 \omega t + 2\alpha \sin \omega t \cos \omega t = 0$$

$$\sin \omega t \left[\frac{\hbar^2}{l} \sin \omega t + \alpha \cos \omega t \right] = 0$$

1^{er} Famille de solutions

$$\text{si } \alpha l = 0 \Rightarrow (\alpha l)_n = n\pi \quad n \in \mathbb{N}^*$$

$$\Rightarrow \alpha_n^2 = \frac{n^2\pi^2}{l^2} = k_n^2 - \left(\frac{\hbar}{l}\right)^2$$

$$k_n^2 = \frac{\hbar^2\pi^2}{l^2} + \left(\frac{\hbar}{l}\right)^2$$

$$\omega_n^2 = \frac{E}{\rho} \int \frac{\partial^2 u}{\partial x^2} + \left(\frac{\hbar}{l}\right)^2 \quad n \in \mathbb{N}^*$$

formes propres

on résout le système linéaire homogène pour $\sin \alpha l = 0$
 $\Rightarrow \cos \alpha l = \pm 1 \neq 0$

$$\Rightarrow A_1 = 0 \quad \text{et} \quad B_2 = B_1$$

on prend arbitrairement $B_2 = 1 \Rightarrow B_1 = 1$

$$X_1(x) = e^{\frac{\hbar^2 x}{l}} \sin \alpha x$$

$$X_2(x) = e^{-\frac{\hbar^2 x}{l}} \sin \alpha x$$

$$X_1(0) = X_2(0) = 0$$

$$X_1(-\alpha) = -X_2(\alpha) \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{ modes propres}$$

2nd Deuxième famille de solutions

$$\frac{\hbar^2}{l} \sin \alpha l + \alpha \cos \alpha l = 0 \Rightarrow \sin \alpha l \neq 0$$

le système linéaire homogène

$$\Rightarrow B_1 = A_2 \cot \alpha l$$

$$B_2 = -A_1 \operatorname{ctan} \alpha l$$

arbitrairement $A_1 = 1$

$$X_1(x) = e^{\frac{\hbar^2 x}{l}} \left[\cos \alpha x + \operatorname{ctan} \alpha l \sin \alpha x \right] = e^{\frac{\hbar^2 x}{l}} \frac{\sin \alpha (l+x)}{\sin \alpha l}$$

$$X_2(x) = e^{-\frac{\hbar^2 x}{l}} \left[\cos \alpha x - \operatorname{ctan} \alpha l \sin \alpha x \right] = e^{-\frac{\hbar^2 x}{l}} \frac{\sin \alpha (l-x)}{\sin \alpha l}$$

$$X_1(0) = X_2(0) = 1$$

$$X_1(-\alpha) = X_2(\alpha) \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{ modes propres.}$$